Inertial Measurement Unit

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1 Introduction

The use of inertial measurement unit (IMU) in robotics has been widely studied and applied in various applications. Especially with the *preintegration* technique [1, 2], IMU measurements can be efficiently fused with other sensor modalities in an optimization-based framework by combining high-frequency inertial measurements between two time instances into a single relative motion constraint.

Preintegration on manifold was first proposed by Forster *et al.* [2], which only considered the manifold structure of the rotation group $SO(3)$. The work was later extended to $SE_2(3)$ for improved consistency and accuracy by [3, 4]. For instance, [3] proposed a preintegration scheme with consideration of the rotating Earth effect, which is important for long-term navigation. A paper by Wang *et al.* [5] introduced an enhanced discretization scheme for preintegration, where the mean is propagated using the exponential function of an automorphism of $SE_2(3)$ and the covariance is propagated using the Magnus expansion. Recently, Delama *et al.* [6] derived a novel discrete-time *equivariant* formulation of the IMU preintegration on the *Special Galilean Group* $Gal(3)$ [7].

In this section, the IMU sensor model is derived based on the work of Wang *et al.* [5] and Brossard *et al.* [3]. *Galilean Group* structure is not considered in this thesis because the uncertainty in time is not modelled.

2 Noise Models

Noise Density

Averaging a noisy signal and outputting it at a lower rate reduces the measured noise of sensor output. *Noise density provides the noise divided by the square root of the sampling rate*. By multiplying the noise density by the square root of the sampling rate, the discrete-time noise standard deviation at that rate can be recovered.

Random Walk

If a noisy output signal from a sensor is integrated, the integration will drift over time due to the noise. By multiplying the random walk by the square root of the time interval, the standard deviation of the drift due to noise can be recovered.

3 Sensor Model

This section discusses sensor modelling for accelerometers and gyroscopes.

3.1 Accelerometer Model

Accelerometers measure specific force $\mathbf{f}_b$, in units of $\mathrm{m\,s^{-2}}$. A simple three-axis accelerometer measurement is [8, § 4.4.6]

$$\tilde{\mathbf{f}}_b(t) = (\mathbf{1} + \mathbf{M}^a)\,\mathbf{f}_b(t) + \boldsymbol{\beta}_b^a(t) + \boldsymbol{\eta}_b^a(t), \tag{1}$$

in which $\mathbf{M}^a \in \mathbb{R}^{3\times 3}$ contains a combination of scale factor and misalignment errors, $\boldsymbol{\beta}_b^a(t)$ is a time-dependent bias, and $\boldsymbol{\eta}_b^a(t)$ is Gaussian white noise. The misalignment matrix is

$$\mathbf{M}^a = \begin{bmatrix} s_1 & m_{12} & m_{13} \\ m_{21} & s_2 & m_{23} \\ m_{31} & m_{32} & s_3 \end{bmatrix}, \tag{2}$$

in which the s_i are the scale factor errors along the i-th axis and where the m_{ij} are the cross-coupling errors between the i-th and j-th axes. The time-dependent bias $\boldsymbol{\beta}_b^a(t)$ is frequently modelled as a continuous-time rate random walk,

$$\dot{\boldsymbol{\beta}}_b^a(t) = \boldsymbol{\eta}_b^{ba}(t), \qquad \boldsymbol{\eta}_b^{ba}(t) \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\mathcal{Q}}^{ba}\delta\left(t-\tau\right)\right), \tag{3}$$

in which $\boldsymbol{\mathcal{Q}}^{ba}$ is the power spectral density (PSD) associated with the rate random walk, in units of $\mathrm{m^2\,s^{-5}}$. Finally, the continuous-time Gaussian white noise is characterized by

$$\boldsymbol{\eta}^a(t) \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\mathcal{Q}}^a\delta\left(t-\tau\right)\right), \tag{4}$$

in which $\boldsymbol{\mathcal{Q}}^a$ is the PSD associated with the white noise, in units of $\mathrm{m^2\,s^{-3}}$. Assuming $\mathbf{M}^a = \mathbf{0}$ and omitting the time argument for brevity, a simplified accelerometer model is

$$\tilde{\mathbf{f}}_b = \mathbf{f}_b + \boldsymbol{\beta}_b^a + \boldsymbol{\eta}_b^a. \tag{5}$$

3.2 Gyroscope Model

Gyroscopes measure angular velocity w.r.t. an inertial frame $\boldsymbol{\omega}_b^{bi}$. A simple three-axis gyroscope measurement is [8, § 4.4.6]

$$\tilde{\boldsymbol{\omega}}_b^{bi}(t) = (\mathbf{1} + \mathbf{M}^g)\,\boldsymbol{\omega}_b^{bi}(t) + \mathbf{B}^g\mathbf{f}_b(t) + \boldsymbol{\beta}_b^g(t) + \boldsymbol{\eta}_b^g(t), \tag{6}$$

in which $\mathbf{M}^g \in \mathbb{R}^{3\times 3}$ contains a combination of scale factor and misalignment errors, $\mathbf{B}^g \in \mathbb{R}^{3\times 3}$ encodes a g-dependent bias induced by the specific force, $\boldsymbol{\beta}_b^g(t)$ is a time-dependent bias, and $\boldsymbol{\eta}_b^g(t)$ is Gaussian white noise. The misalignment matrices $\mathbf{M}^g$ and $\mathbf{B}^g$ have a similar structure to (2). The time-dependent bias $\boldsymbol{\beta}_b^g(t)$ is frequently modelled as a continuous-time rate random walk,

$$\dot{\boldsymbol{\beta}}_b^g(t) = \boldsymbol{\eta}_b^{bg}(t), \qquad \boldsymbol{\eta}_b^{bg}(t) \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\mathcal{Q}}^{bg}\delta\left(t-\tau\right)\right), \tag{7}$$

in which $\boldsymbol{\mathcal{Q}}^{bg}$ is the power spectral density (PSD) associated with the rate random walk, in units of $\mathrm{rad^2\,s^{-3}}$. Finally, the continuous-time Gaussian white noise is characterized by

$$\boldsymbol{\eta}^g(t) \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\mathcal{Q}}^g\delta\left(t-\tau\right)\right), \tag{8}$$

in which $\boldsymbol{\mathcal{Q}}^g$ is the PSD associated with the white noise, in units of $\mathrm{rad^2\,s^{-1}}$. Assuming $\mathbf{M}^g = \mathbf{B}^g = \mathbf{0}$ and omitting the time argument for brevity, a simplified gyroscope model is

$$\tilde{\boldsymbol{\omega}}_b^{bi} = \boldsymbol{\omega}_b^{bi} + \boldsymbol{\beta}_b^g + \boldsymbol{\eta}_b^g. \tag{9}$$

4 Kinematics

This section derives the continuous-time kinematics of a vehicle on the rotating Earth as a function of the IMU measurements. The continuous-time kinematics of a vehicle on the rotating Earth is given as

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\left(\boldsymbol{\omega}_b^{bi}\right)^{\times} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{C}_{ab}, \tag{10a}$$

$$\dot{\mathbf{v}}_a^{zo/a} = \mathbf{a}_a^{zo/i/i} - 2\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zo/a} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{r}_a^{zo}, \tag{10b}$$

$$\dot{\mathbf{r}}_a^{zo} = \mathbf{v}_a^{zo/a}. \tag{10c}$$

Replacing the velocity kinematics with a specific force model yields

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\left(\boldsymbol{\omega}_b^{bi}\right)^{\times} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{C}_{ab}, \tag{11a}$$

$$\dot{\mathbf{v}}_a^{zo/a} = \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) - 2\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zo/a}, \tag{11b}$$

$$\dot{\mathbf{r}}_a^{zo} = \mathbf{v}_a^{zo/a}. \tag{11c}$$

Now, (11) is composed of the specific force $\mathbf{f}_b$ and the angular rate $\boldsymbol{\omega}_b^{bi}$ measured by the accelerometer and gyroscope, respectively. However, the given kinematics equations are highly nonlinear and are difficult to realize in a compact form. The following section presents a realization of a compact form of the continuous-time kinematics equations that can be solved using the differential Sylvester equation.

4.1 Compact Form

Note that $\underrightarrow{v}^{zo/a} = \underrightarrow{v}^{zw/a} + \underrightarrow{v}^{wo/a}$. The time rate of change of a vector from $\underrightarrow{r}^{wo}$ w.r.t. the local navigation frame $\mathcal{F}_a$ is $\underrightarrow{0}$ resulting in $\underrightarrow{v}^{zo/a} = \underrightarrow{v}^{zw/a}$. Thus, (11b) and (11c) can be rewritten as

$$\dot{\mathbf{v}}_a^{zw/a} = \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) - 2\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/a}, \tag{12a}$$

$$\dot{\mathbf{r}}_a^{zw} = \mathbf{v}_a^{zw/a}. \tag{12b}$$

Now, consider the auxiliary velocity $\underrightarrow{v}^{zw/i} = \underrightarrow{v}^{zw/a} + \underrightarrow{\omega}^{ai} \times \underrightarrow{r}^{zw}$ introduced in [3]. The time derivative of the auxiliary velocity w.r.t. frame $\mathcal{F}_a$ is

$$\left(\underrightarrow{v}^{zw/i}\right)^{\bullet a} = \left(\underrightarrow{v}^{zw/a}\right)^{\bullet a} + \cancelto{\underrightarrow{0}}{\left(\underrightarrow{\omega}^{ai}\right)^{\bullet a}} \times \underrightarrow{r}^{zw} + \underrightarrow{\omega}^{ai} \times \left(\underrightarrow{r}^{zw}\right)^{\bullet a} \tag{13a}$$

$$= \left(\underrightarrow{v}^{zw/a}\right)^{\bullet a} + \underrightarrow{\omega}^{ai} \times \underrightarrow{v}^{zw/a}. \tag{13b}$$

With slight abuse of notation, $\underrightarrow{\mathcal{F}}_a^{\mathsf{T}}\dot{\mathbf{v}}_a^{zw/i} = (\underrightarrow{v}^{zw/i})^{\bullet a}$, resolving the above equations in $\mathcal{F}_a$ yields

$$\dot{\mathbf{v}}_a^{zw/i} = \dot{\mathbf{v}}_a^{zw/a} + \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/a} \tag{14a}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) - 2\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/a} + \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/a} \tag{14b}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/a} \tag{14c}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\mathbf{v}_a^{zw/i} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{r}_a^{zw}\right) \tag{14d}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) + \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{r}_a^{zw} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/i} \tag{14e}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi) + \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\mathbf{r}_a^{zo} - \mathbf{r}_a^{wo}\right) - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/i}. \tag{14f}$$

Substituting gravitational acceleration into the above equation yields

$$\dot{\mathbf{v}}_a^{zw/i} = \mathbf{C}_{ab}\mathbf{f}_b + \tilde{\mathbf{g}}_a - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{r}_a^{zo} + \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\mathbf{r}_a^{zo} - \mathbf{r}_a^{wo}\right) - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/i} \tag{15a}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \tilde{\mathbf{g}}_a - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{r}_a^{wo} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/i} \tag{15b}$$

$$= \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi_w) - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/i}. \tag{15c}$$

Note that the gravity vector $\mathbf{g}_a(\phi_w)$ is now a function of the latitude at the reference point ϕ_w, and it is computed only once. However, the Earth rotation in the local navigation frame $\boldsymbol{\omega}_a^{ai} = \mathbf{C}_{ae}(\phi, \lambda)\boldsymbol{\omega}_e^{ai}$ is still a function of the latitude and longitude at a given time instance. Thus, in practice, the Earth rotation is updated along with the vehicle position.

The continuous-time kinematics equations can be rewritten as

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\left(\boldsymbol{\omega}_b^{bi}\right)^{\times} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{C}_{ab}, \tag{16a}$$

$$\dot{\mathbf{v}}_a^{zw/i} = \mathbf{C}_{ab}\mathbf{f}_b + \mathbf{g}_a(\phi_w) - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{v}_a^{zw/i}, \tag{16b}$$

$$\dot{\mathbf{r}}_a^{zw} = \mathbf{v}_a^{zw/i} - \left(\boldsymbol{\omega}_a^{ai}\right)^{\times}\mathbf{r}_a^{zw}. \tag{16c}$$

In a compact form, (16) can be written as

$$\dot{\mathbf{X}}_{ab}^{zw} = \mathbf{G}\mathbf{X}_{ab}^{zw} + \mathbf{X}_{ab}^{zw}\mathbf{U} \tag{17a}$$

$$= \begin{bmatrix} -\left(\boldsymbol{\omega}_a^{ai}\right)^{\times} & \mathbf{g}_a^w & \mathbf{0} \\ \mathbf{0} & 0 & -1 \\ \mathbf{0} & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{C}_{ab} & \mathbf{v}_a^{zw/i} & \mathbf{r}_a^{zw} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{ab} & \mathbf{v}_a^{zw/i} & \mathbf{r}_a^{zw} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}\begin{bmatrix} \left(\boldsymbol{\omega}_b^{bi}\right)^{\times} & \mathbf{f}_b & \mathbf{0} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix}, \tag{17b}$$

where $\mathbf{g}_a^w = \mathbf{g}_a(\phi_w)$, $\mathbf{X}_{ab}^{zw} \in SE_2(3)$ is the state of the vehicle, $\mathbf{G} \in \mathbb{R}^{5\times5}$ is the gravity matrix, and $\mathbf{U} \in \mathbb{R}^{5\times5}$ is the input matrix.

The compact continuous-time kinematics from (17) has a form of a *differential Sylvester equation* [9]. Assuming the input matrix at t_{k-1} is constant over the integration period $\Delta t = t_k - t_{k-1}$, the solution to the differential Sylvester equation with initial condition $\mathbf{X}_0 = (\mathbf{X}_{ab}^{zw})_{k-1}$ is

$$(\mathbf{X}_{ab}^{zw})_k = \exp(\Delta t\mathbf{G})(\mathbf{X}_{ab}^{zw})_{k-1}\exp(\Delta t\mathbf{U}(t_{k-1})) \tag{18a}$$

$$= \mathbf{G}_{k-1}(\mathbf{X}_{ab}^{zw})_{k-1}\mathbf{U}_{k-1}. \tag{18b}$$

The detailed derivation of the discrete-time $\mathbf{G}_{k-1}$ and $\mathbf{U}_{k-1}$ are in Appendix A and Appendix B, respectively. For clarity of notation, recall that the time argument t has been omitted from the states and the measurements as $\mathbf{C}_{ab} = \mathbf{C}_{ab}(t)$, $\tilde{\boldsymbol{\omega}}_b^{ba} = \tilde{\boldsymbol{\omega}}_b^{ba}(t)$, and $\mathbf{f}_b = \mathbf{f}_b(t)$. Therefore, to indicate a state at some arbitrary time t_k a subscript is added to the vehicle/sensor frame and datum. For example, $\mathbf{C}_{ab_k} = \mathbf{C}_{ab}(t_k)$, $\mathbf{v}_a^{z_kw/i} = \mathbf{v}_a^{zw/i}(t_k)$, $\mathbf{r}_a^{z_kw} = \mathbf{r}_a^{zw}(t_k)$, $\boldsymbol{\beta}_{b_k}^g = \boldsymbol{\beta}_b^g(t_k)$, and $\boldsymbol{\eta}_{b_k}^g = \boldsymbol{\eta}_b^g(t_k)$. Also for further derivations, the subscripts and the superscripts of the states are omitted unless necessary. For example, $\mathbf{C}_k = \mathbf{C}_{ab_k}$, $\mathbf{v}_k = \mathbf{v}_a^{z_kw/i}$, $\mathbf{r}_k = \mathbf{r}_a^{z_kw}$, and $\mathbf{X}_k = (\mathbf{X}_{ab}^{zw})_k$.

4.2 Input Matrix

The components of the input matrix are simply a combination of a skew-symmetric matrix of an angular velocity and the specific force, which are the measurements from the gyroscope and accelerometer, respectively. Thus, substituting the gyroscope and accelerometer models, (6) and (1), respectively, in $\mathbf{U}$ yields

$$\mathbf{U} = \begin{bmatrix} \left((\mathbf{1}+\mathbf{M}^g)^{-1}\left(\tilde{\boldsymbol{\omega}}_b^{bi} - \boldsymbol{\beta}_b^g - \boldsymbol{\eta}_b^g - \mathbf{B}^g\mathbf{f}_b\right)\right)^{\times} & (\mathbf{1}+\mathbf{M}^a)^{-1}\left(\tilde{\mathbf{f}}_b - \boldsymbol{\beta}_b^a - \boldsymbol{\eta}_b^a\right) & \mathbf{0} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix}. \tag{19}$$

Neglecting the misalignment error matrices $\mathbf{M}^g = \mathbf{M}^a = \mathbf{B}^g = \mathbf{0}$, the input matrix simplifies to

$$\mathbf{U} = \begin{bmatrix} \left(\tilde{\boldsymbol{\omega}}_b^{bi} - \boldsymbol{\beta}_b^g - \boldsymbol{\eta}_b^g\right)^{\times} & \tilde{\mathbf{f}}_b - \boldsymbol{\beta}_b^a - \boldsymbol{\eta}_b^a & \mathbf{0} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix} = \mathbf{D} + \tilde{\mathbf{U}} - \mathbf{B} - \mathbf{N}, \tag{20}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{U}} = \begin{bmatrix} \tilde{\mathbf{u}} \\ \mathbf{0} \end{bmatrix}^{\wedge} \in \mathfrak{se}_2(3), \qquad \mathbf{B} = \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{0} \end{bmatrix}^{\wedge} \in \mathfrak{se}_2(3), \qquad \mathbf{N} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix}^{\wedge} \in \mathfrak{se}_2(3), \tag{21}$$

$\tilde{\mathbf{u}}^\mathsf{T} = \begin{bmatrix}\tilde{\boldsymbol{\omega}}_b^{bi\,\mathsf{T}} & \tilde{\mathbf{f}}_b^{\,\mathsf{T}}\end{bmatrix}$, $\boldsymbol{\beta}^\mathsf{T} = \begin{bmatrix}\boldsymbol{\beta}^{g\,\mathsf{T}} & \boldsymbol{\beta}^{a\,\mathsf{T}}\end{bmatrix}$, and $\boldsymbol{\eta}^\mathsf{T} = \begin{bmatrix}\boldsymbol{\eta}^{g\,\mathsf{T}} & \boldsymbol{\eta}^{a\,\mathsf{T}}\end{bmatrix}$. The matrix $\mathbf{D}$ is a constant matrix that is referred to as a "time machine" in [10, § 9.4.7].

Taking the matrix exponential of the input matrix at t_k over the integration period Δt is

$$\begin{aligned}\mathbf{U}_k &= \exp\left(\Delta t \mathbf{U}(t_k)\right) && \text{(22a)}\\ &= \begin{bmatrix}\exp\left(\Delta t \boldsymbol{\omega}_k^\times\right) & \Delta t \mathbf{J}_\ell\left(\Delta t \boldsymbol{\omega}_k\right)\mathbf{f}_k & \frac{1}{2}\Delta t^2 \mathbf{N}\left(\Delta t \boldsymbol{\omega}_k\right)\mathbf{f}_k \\ \mathbf{0} & 1 & \Delta t \\ \mathbf{0} & 0 & 1\end{bmatrix}. && \text{(22b)}\end{aligned}$$

This discretized input matrix can be decomposed into a constant part and a time-varying part as

$$\mathbf{U}_k = \mathbf{D}_k \mathbf{U}_k' = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \Delta t \\ \mathbf{0} & 0 & 1\end{bmatrix} \begin{bmatrix}\exp\left(\Delta t \boldsymbol{\omega}_k^\times\right) & \Delta t \mathbf{J}_\ell\left(\Delta t \boldsymbol{\omega}_k\right)\mathbf{f}_k & \frac{1}{2}\Delta t^2 \mathbf{N}\left(\Delta t \boldsymbol{\omega}_k\right)\mathbf{f}_k \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1\end{bmatrix}, \tag{23}$$

where $\mathbf{D}_k = \exp(\mathbf{D}\Delta t)$ and $\mathbf{U}_k' \in SE_2(3)$. The decomposition will be useful for further derivation.

5 Preintegration

This section is an extension to [3, 5] and [10, §9.4.8]. The high-frequency IMU measurements can be integrated between two timestamps such that the states can be estimated at a lower frequency. The compact form in (18) can easily propagate multiple states from some arbitrary time t_i to t_j

$$\begin{aligned}\mathbf{X}_j &= \left(\prod_{k=i}^{j-1} \mathbf{G}_k\right) \mathbf{X}_i \left(\prod_{k=i}^{j-1} \mathbf{U}_k\right) && \text{(24a)}\\ &= \mathbf{G}_{ij} \mathbf{X}_i \mathbf{U}_{ij}, && \text{(24b)}\end{aligned}$$

where the term $\mathbf{G}_{ij}$ is constant and does not depend on either state or measurements. The derivation of $\mathbf{G}_{ij}$ is shown in Appendix C. The term $\mathbf{U}_{ij}$ is referred to as *relative motion increment* (RMI) generated from multiple IMU measurements. Preintegration is the process of computing the RMI from multiple IMU measurements and propagating its uncertainty. The following sections discuss the error dynamics of the RMI and its noise propagation.

5.1 Continuous-time RMI Error Dynamics

The RMI dynamics from t_i to some arbitrary time t_k is given by

$$\mathbf{U}_{ik} = \mathbf{U}_{ii} \prod_{\tau=i}^{k-1} \mathbf{U}_\tau = \mathbf{U}_{ii} \prod_{\tau=i}^{k-1} \exp\left(\mathbf{U}(t_\tau \Delta t)\right) = \begin{bmatrix}\Delta \mathbf{C}_{ik} & \Delta \mathbf{v}_{ik} & \Delta \mathbf{r}_{ik} \\ \mathbf{0} & 1 & \Delta t_{ik} \\ \mathbf{0} & 0 & 1\end{bmatrix}, \tag{25}$$

where $\mathbf{U}_{ii} = \mathbf{1}$ is the initial condition of the RMI at t_i. This discretely integrates the input matrix over the integration period Δt from t_i to t_k. However, the continuous-time RMI kinematics,

$$\mathbf{U}_{ik+1} = \mathbf{U}_{ii} \exp\left(\int_{t_i}^{t_k} \mathbf{U}(\tau)\mathrm{d}\tau\right), \tag{26}$$

integrates the continuous input function. Denoting $\boldsymbol{\Xi}(t_i, t_k) = \mathbf{U}_{ik+1} = \mathbf{D}(t_i, t_k)\boldsymbol{\Xi}'(t_i, t_k)$, where $\boldsymbol{\Xi}'(t_i, t_k) \in SE_2(3)$, the time derivative of the above equation is then

$$\dot{\boldsymbol{\Xi}}(t_i, t) = \boldsymbol{\Xi}(t_i, t)\mathbf{U}(t). \tag{27}$$

The time argument t is omitted for brevity unless necessary for the following derivations. The error of the RMI is defined in two ways, either by right multiplication or left multiplication, as

$$\begin{aligned}\delta \boldsymbol{\Xi}^R &= \bar{\boldsymbol{\Xi}}^{-1} \boldsymbol{\Xi}, && \text{(28a)}\\ \delta \boldsymbol{\Xi}^L &= \boldsymbol{\Xi} \bar{\boldsymbol{\Xi}}^{-1}. && \text{(28b)}\end{aligned}$$

The time-derivative of the RMI error by right multiplication is given as

$$\begin{aligned}
\delta\dot{\Xi}^R &= \dot{\bar{\Xi}}^{-1}\Xi + \bar{\Xi}^{-1}\dot{\Xi} && \text{(29a)}\\
&= -\bar{\Xi}^{-1}\dot{\bar{\Xi}}\bar{\Xi}^{-1}\Xi + \bar{\Xi}^{-1}\dot{\Xi} && \text{(29b)}\\
&= -\bar{\Xi}^{-1}\left(\bar{\Xi}\left(\mathbf{D}+\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\right)\bar{\Xi}^{-1}\Xi + \bar{\Xi}^{-1}\left(\Xi\mathbf{U}\right) && \text{(29c)}\\
&= -\left(\mathbf{D}+\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\delta\Xi + \delta\Xi\left(\mathbf{D}+\tilde{\mathbf{U}}-\mathbf{B}-\mathbf{N}\right) && \text{(29d)}\\
\approx \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{1}+\delta\mathbf{u}^\wedge\right) &= -\left(\mathbf{D}+\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\left(\mathbf{1}+\delta\mathbf{u}^\wedge\right) + \left(\mathbf{1}+\delta\mathbf{u}^\wedge\right)\left(\mathbf{D}+\tilde{\mathbf{U}}-\bar{\mathbf{B}}-\delta\mathbf{B}-\delta\mathbf{N}\right) && \text{(29e)}\\
&= -\mathbf{D}\delta\mathbf{u}^\wedge + \delta\mathbf{u}^\wedge\mathbf{D} - \left(\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\delta\mathbf{u}^\wedge + \delta\mathbf{u}^\wedge\left(\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right) - \delta\mathbf{B} - \delta\mathbf{N} + \text{H.O.T} && \text{(29f)}\\
&\approx \left(-\mathbf{ad}\left(\mathbf{D}\right)\delta\mathbf{u}\right)^\wedge + \left(-\mathbf{ad}\left(\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\delta\mathbf{u}\right)^\wedge - \delta\mathbf{B} - \delta\mathbf{N}, && \text{(29g)}\\
\delta\dot{\mathbf{u}} &= \left(-\mathbf{ad}\left(\mathbf{D}\right) - \mathbf{ad}\left(\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\right)\delta\mathbf{u} - \delta\mathbf{B}^\vee - \delta\mathbf{N}^\vee, && \text{(29h)}
\end{aligned}$$

where $\mathbf{ad}(\mathbf{D})$, the adjoint matrix of the time-machine matrix, is given in Appendix D.

The time-derivative of the RMI error by left multiplication is given as

$$\begin{aligned}
\delta\dot{\Xi}^L &= \dot{\Xi}\bar{\Xi}^{-1} + \Xi\dot{\bar{\Xi}}^{-1} && \text{(30a)}\\
&= \dot{\Xi}\bar{\Xi}^{-1} - \Xi\bar{\Xi}^{-1}\dot{\bar{\Xi}}\bar{\Xi}^{-1} && \text{(30b)}\\
&= \Xi\mathbf{U}\bar{\Xi}^{-1} - \Xi\bar{\Xi}^{-1}\bar{\Xi}\left(\mathbf{D}+\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right)\bar{\Xi}^{-1} && \text{(30c)}\\
&= \delta\Xi\bar{\Xi}\left(-\delta\mathbf{B}-\delta\mathbf{N}\right)\bar{\Xi}^{-1} && \text{(30d)}\\
\approx \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{1}+\delta\mathbf{u}^\wedge\right) &= \left(\mathbf{1}+\delta\mathbf{u}^\wedge\right)\bar{\Xi}\left(-\delta\mathbf{B}-\delta\mathbf{N}\right)\bar{\Xi}^{-1} && \text{(30e)}\\
&= \bar{\Xi}\left(-\delta\mathbf{B}-\delta\mathbf{N}\right)\bar{\Xi}^{-1} + \text{H.O.T} && \text{(30f)}\\
&\approx \mathbf{D}(t_i,t)\bar{\Xi}'(t_i,t)\left(-\delta\mathbf{B}-\delta\mathbf{N}\right)\bar{\Xi}'^{-1}(t_i,t)\mathbf{D}^{-1}(t_i,t) && \text{(30g)}\\
&= \mathbf{D}(t_i,t)\left(\mathbf{Ad}\left(\bar{\Xi}'(t_i,t)\right)\left(-\delta\mathbf{B}-\delta\mathbf{N}\right)^\vee\right)^\wedge\mathbf{D}^{-1}(t_i,t) && \text{(30h)}\\
&= \left(\mathbf{Ad}\left(\mathbf{D}(t_i,t)\right)\mathbf{Ad}\left(\bar{\Xi}'(t_i,t)\right)\left(-\delta\mathbf{B}-\delta\mathbf{N}\right)^\vee\right)^\wedge, && \text{(30i)}\\
\delta\dot{\mathbf{u}} &= -\mathbf{Ad}\left(\mathbf{D}(t_i,t)\right)\mathbf{Ad}\left(\bar{\Xi}'(t_i,t)\right)\delta\mathbf{B}^\vee - \mathbf{Ad}\left(\mathbf{D}(t_i,t)\right)\mathbf{Ad}\left(\bar{\Xi}'(t_i,t)\right)\delta\mathbf{N}^\vee, && \text{(30j)}
\end{aligned}$$

where $\mathbf{Ad}(\mathbf{D}(t_i,t))$ is given in Appendix D, and $\bar{\Xi}(t_i,t) \approx \prod_{k=i}^{t-1}\bar{\mathbf{U}}_k$ is the accumulated relative motion increment from $\Xi(t_i,t_i) = \mathbf{1}$ to time t.

5.2 Incorporating Bias

In practice, biases are not observable when dead-reckoning with an IMU. Thus, the gyroscope and accelerometer biases are assumed to be constant over the integration period. Despite the constant mean of the biases, their uncertainties grow over time. Thus, the first-order approximation of the continuous-time random walk biases are compounded to the RMI error dynamics as

$$\begin{bmatrix}\delta\dot{\mathbf{u}}\\ \delta\dot{\boldsymbol{\beta}}\end{bmatrix} = \begin{bmatrix}-\mathbf{ad}\left(\mathbf{D}\right)-\mathbf{ad}\left(\tilde{\mathbf{U}}-\bar{\mathbf{B}}\right) & -\mathbf{H}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\mathbf{u}\\ \delta\boldsymbol{\beta}\end{bmatrix} + \begin{bmatrix}-\mathbf{H} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\eta}\\ \delta\boldsymbol{\eta}^b\end{bmatrix} \quad \text{(right)}, \tag{31a}$$

$$\begin{bmatrix}\delta\dot{\mathbf{u}}\\ \delta\dot{\boldsymbol{\beta}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{Ad}\left(\bar{\Xi}(t_0,t)\right)\mathbf{H}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\mathbf{u}\\ \delta\boldsymbol{\beta}\end{bmatrix} + \begin{bmatrix}-\mathbf{Ad}\left(\bar{\Xi}(t_0,t)\right)\mathbf{H} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\eta}\\ \delta\boldsymbol{\eta}^b\end{bmatrix} \quad \text{(left)}, \tag{31b}$$

where $\mathbf{Ad}(\bar{\Xi}(t_0,t)) = \mathbf{Ad}(\mathbf{D}(t_0,t))\mathbf{Ad}(\bar{\Xi}'(t_0,t))$, and

$$\mathbf{H} = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\end{bmatrix}^\mathsf{T}. \tag{32}$$

5.3 Covariance Propagation

The continuous-time RMI error dynamics can be discretized using Van Loan's method [11]. Given a continuous-time linear system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{L}(t)\boldsymbol{\eta}(t), \tag{33}$$

where $\boldsymbol{\eta}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}(t))$ is a white Gaussian noise. The corresponding discrete-time system is

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \boldsymbol{\eta}_k^d, \quad \boldsymbol{\eta}_k^d \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k^d), \tag{34}$$

where $\mathbf{A}_k$ and $\mathbf{Q}_k^d$ are the discrete-time state transition matrix and process noise covariance matrix, respectively. Assuming that the system is time-invariant over the interval $[t_k, t_{k+1}]$, these matrices can be computed by taking the matrix exponential of the following block matrix

$$\exp\left(\begin{bmatrix}\mathbf{A}(t_k) & \mathbf{L}(t_k)\mathbf{Q}(t_k)\mathbf{L}^\mathsf{T}(t_k) \\ \mathbf{0} & -\mathbf{A}^\mathsf{T}(t_k)\end{bmatrix}\Delta t\right) = \begin{bmatrix}\boldsymbol{\Upsilon}_k^{11} & \boldsymbol{\Upsilon}_k^{12} \\ \mathbf{0} & \boldsymbol{\Upsilon}_k^{22}\end{bmatrix}, \tag{35}$$

and the discrete-time matrices are then given as

$$\mathbf{A}_k = \boldsymbol{\Upsilon}_k^{11}, \tag{36a}$$

$$\mathbf{Q}_k^d = \boldsymbol{\Upsilon}_k^{12}\mathbf{A}_k^\mathsf{T}. \tag{36b}$$

This results in an RMI covariance to be propagated recursively by

$$\mathbf{Q}_{ik+1} = \mathbf{A}_k\mathbf{Q}_{ik}\mathbf{A}_k^\mathsf{T} + \mathbf{Q}_k^d, \tag{37}$$

which is initialized with $\mathbf{Q}_{ii} = \mathbf{0}$. Also, the RMI Jacobian is propagated recursively from $\mathbf{J}_{ii} = \mathbf{1}$ as

$$\mathbf{J}_{ik+1} = \mathbf{A}_k\mathbf{J}_{ik}. \tag{38}$$

6 IMU Factor

There are two approaches to fusing IMU measurements in a continuous-time framework: preintegration and *gyro as a measurement*.

6.1 Preintegration Factor

Recall the IMU propagation model in (24) that involves the preintegration. The preintegrated IMU residual for two subsequent states at t_i and t_j is defined as

$$\mathbf{e}_{ij}^I(\mathcal{X}) = \begin{bmatrix}{}^{(\cdot)}\left(\tilde{\mathbf{U}}_{ij}^+ \ominus \check{\mathbf{U}}_{ij}(\mathbf{X}_i, \mathbf{X}_j)\right) \\ \boldsymbol{\beta}_j - \boldsymbol{\beta}_i\end{bmatrix}, \tag{39a}$$

$$\tilde{\mathbf{U}}_{ij}^+ = \tilde{\mathbf{U}}_{ij} \oplus {}^{(\cdot)}\frac{D\tilde{\mathbf{U}}_{ij}}{D\boldsymbol{\beta}_i}\delta\boldsymbol{\beta}_i, \tag{39b}$$

where $(\cdot)$ denotes the right- or left-operators, $\tilde{\mathbf{U}}_{ij}$ is the preintegrated IMU measurement from (25), $D\tilde{\mathbf{U}}_{ij}/D\boldsymbol{\beta}_i$ is a sub-block of (38), the Jacobian of the preintegrated IMU measurement with respect to the IMU biases, $\delta\boldsymbol{\beta}_i = \boldsymbol{\beta}_i - \bar{\boldsymbol{\beta}}_i$ is the difference between the current bias and the bias used for preintegration, and $\check{\mathbf{U}}_{ij}(\mathbf{X}_i, \mathbf{X}_j) = (\mathbf{G}_{ij}\mathbf{X}_i)^{-1}\mathbf{X}_j$. Detailed derivations of the preintegrated IMU Jacobian are provided in Appendix E.

6.2 Gyro as a Measurement Factor

Another way of using the IMU is to treat it as a high-frequency measurement proposed by [12]. Recall the simplified gyroscope model (9). Since the angular velocity is part of the state, the rate gyro measurement can be directly related to the angular velocity state,

$$\mathbf{e}_\tau^{Ig}(\mathcal{X}) = \tilde{\boldsymbol{\omega}}_{b_\tau}^{b_\tau a} - \boldsymbol{\omega}_b^{ba}(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau) - \boldsymbol{\beta}_b^g(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau), \qquad t_i < \tau < t_j, \tag{40}$$

where $\tilde{\boldsymbol{\omega}}_{b_\tau}^{b_\tau a}$ is the measured angular velocity at time τ, and $\boldsymbol{\omega}_b^{ba}(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau)$ and $\boldsymbol{\beta}_b^{g}(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau)$ represent the angular velocity and bias states queried at time τ, respectively. While the angular velocity state is queried using GP interpolation, the bias state is queried using linear interpolation,

$$\boldsymbol{\beta}(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau) = (1-\alpha)\boldsymbol{\beta}_i + \alpha\boldsymbol{\beta}_j, \qquad \alpha = \frac{\tau - t_i}{t_j - t_i},\ t_i \leq \tau \leq t_j. \tag{41}$$

The acceleration, however, is not part of the state space, so the acceleration measurement cannot be directly related to the state variables. The accelerometer measurements are, therefore, preintegrated to get the change in velocity, and the change in velocity can be related to the state variables. Recall the discrete-time IMU kinematics with pre-integration (24),

$$\mathbf{T}_j = \mathbf{G}_{ij}\mathbf{T}_i\mathbf{U}_{ij}.$$

Neglecting the Earth's rotation and the position columns in the above equation yields

$$\underbrace{\begin{bmatrix} \mathbf{C}_{ab_j} & \mathbf{v}_a^{z_j w/a} \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{T}'_j} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{g}_a^{w} \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{G}'_{ij}} \underbrace{\begin{bmatrix} \mathbf{C}_{ab_i} & \mathbf{v}_a^{z_i w/a} \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{T}'_i} \underbrace{\begin{bmatrix} \Delta\mathbf{C}_{ij} & \Delta\mathbf{v}_{ij} \\ \mathbf{0} & 1 \end{bmatrix}}_{\mathbf{U}'_{ij}}, \tag{42}$$

where $\mathbf{T}'_i$, $\mathbf{G}'_{ij}$, $\mathbf{T}'_j$, and $\mathbf{U}'_{ij}$ have the same structure as elements of $SE(3)$. In [13], the author calls this a *homogeneous Galilean* group, $HG(3)$, to avoid confusion with the special Euclidean group, $SE(3)$ that consists of a rotation and translation instead of a rotation and velocity. However, since $HG(3)$ and $SE(3)$ are isomorphic, $SE(3)$ is used in this thesis. The input matrix $\mathbf{U}'_{ij}$ can be computed similarly to the RMI described in Section 5 by compounding the input matrix $\mathbf{U}'_\tau$ at each time step, which is given as

$$\mathbf{U}'_{ij} = \prod_{\tau=i}^{j-1} \mathbf{U}'_\tau = \prod_{\tau=i}^{j-1} \operatorname{Exp}\left(\mathbf{u}'_\tau \delta t\right), \tag{43}$$

where $\mathbf{u}'^{\mathsf{T}}_\tau = \begin{bmatrix} \boldsymbol{\omega}_{b_\tau}^{b_\tau a \mathsf{T}} & (\mathbf{f}_{b_\tau} - \boldsymbol{\beta}_{b_\tau}^{a})^{\mathsf{T}} \end{bmatrix}$. Since the angular velocity is part of the state space, the input matrix now consists of the state variable. Given that the preintegration of the acceleration measurement is a function of the angular velocity state, accelerometer measurement, and the accelerometer bias state at each time step τ, the residual can then be formulated as

$$\mathbf{e}_{ij}^{Ia}(\mathcal{X}) = \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix}^{(\cdot)} \left(\mathbf{U}'_{ij} \ominus \check{\mathbf{U}}'_{ij}\right), \tag{44a}$$

$$\check{\mathbf{U}}'_{ij} = \left(\mathbf{G}'_{ij}\mathbf{T}'_i\right)^{-1}\mathbf{T}'_j, \tag{44b}$$

where $(\cdot)$ denotes the right- or left-operators. The Jacobian of the residuals $\mathbf{e}_\tau^{Ig}$ and $\mathbf{e}_\tau^{Ia}$ with respect to the state is derived in Appendix E.2. With the random walk model for the biases,

$$\mathbf{e}_{ij}^{Ib}(\mathcal{X}) = \boldsymbol{\beta}_j - \boldsymbol{\beta}_i, \tag{45}$$

the IMU cost function J^I is now

$$J^I = \|\mathbf{e}_{ii+1}^{Ia}(\mathcal{X})\|^2_{\boldsymbol{\Sigma}_{\mathcal{I}'_{ii+1}}} + \|\mathbf{e}_{ii+1}^{Ib}(\mathcal{X})\|^2_{\boldsymbol{\Sigma}_{\mathcal{B}_{ii+1}}} + \sum_{\tau \in \mathcal{I}_{ii+1}} \|\mathbf{e}_\tau^{Ig}(\mathcal{X})\|^2_{\boldsymbol{\Sigma}_{\mathcal{I}_\tau}}, \tag{46}$$

where $\boldsymbol{\Sigma}_{\mathcal{I}'_{ii+1}}$, $\boldsymbol{\Sigma}_{\mathcal{B}_{ii+1}}$, and $\boldsymbol{\Sigma}_{\mathcal{I}_\tau}$ are the covariance of the preintegrated acceleration measurement, the bias random walk, and the gyroscope measurement, respectively.

The major drawback of this approach is that during the preintegration of the acceleration measurement, the angular velocity and the accelerometer bias state are queried at each measurement time step, which defeats the purpose of preintegration. This problem could be solved by using the white-noise-on-jerk (WNOJ) motion prior, but this would require the angular acceleration to be part of the state space, which is not observable.

A Discrete-time Gravity Matrix

The discrete-time $\mathbf{G}_{k-1}$, with $\boldsymbol{\varphi} = \boldsymbol{\omega}_a^{ai}$ and $\Delta t = t_k - t_{k-1}$, is

$$
\begin{aligned}
\mathbf{G}_{k-1} &= \exp\left(\Delta t \mathbf{G}\right) && \text{(47a)}\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\Delta t \mathbf{G}\right)^k && \text{(47b)}\\
&= \mathbf{1} + \Delta t \mathbf{G} + \frac{1}{2}\left(\Delta t \mathbf{G}\right)^2 + \frac{1}{6}\left(\Delta t \mathbf{G}\right)^3 + \frac{1}{24}\left(\Delta t \mathbf{G}\right)^4 + \dots && \text{(47c)}\\
&= \mathbf{1} + \Delta t \begin{bmatrix} -\boldsymbol{\varphi}^{\times} & \mathbf{g} & \\ & & -1 \\ & & \end{bmatrix} + \frac{\Delta t^2}{2}\begin{bmatrix} \left(\boldsymbol{\varphi}^{\times}\right)^2 & -\boldsymbol{\varphi}^{\times}\mathbf{g} & -\mathbf{g} \\ & & \\ & & \end{bmatrix} \\
&\quad + \frac{\Delta t^3}{6}\begin{bmatrix} -\left(\boldsymbol{\varphi}^{\times}\right)^3 & \left(\boldsymbol{\varphi}^{\times}\right)^2\mathbf{g} & \boldsymbol{\varphi}^{\times}\mathbf{g} \\ & & \\ & & \end{bmatrix} + \frac{\Delta t^4}{24}\begin{bmatrix} \left(\boldsymbol{\varphi}^{\times}\right)^4 & -\left(\boldsymbol{\varphi}^{\times}\right)^3\mathbf{g} & -\left(\boldsymbol{\varphi}^{\times}\right)^2\mathbf{g} \\ & & \\ & & \end{bmatrix} + \dots && \text{(47d)}\\
&= \begin{bmatrix} \mathbf{G}_{k-1}^1 & \mathbf{g}_{kk-1}^2 & \mathbf{g}_{kk-1}^3 \\ & 1 & -\Delta t \\ & & 1 \end{bmatrix}. && \text{(47e)}
\end{aligned}
$$

Note that $\mathbf{G}$ is time invariant. Taking each term in the expansion of $\mathbf{G}_{k-1}$ yields

$$
\begin{aligned}
\mathbf{G}_{k-1}^1 &= 1 - \Delta t \boldsymbol{\varphi}^{\times} + \frac{1}{2}\left(\Delta t \boldsymbol{\varphi}^{\times}\right)^2 - \frac{1}{6}\left(\Delta t \boldsymbol{\varphi}^{\times}\right)^3 + \frac{1}{24}\left(\Delta t \boldsymbol{\varphi}^{\times}\right)^4 + \dots\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\Delta t \boldsymbol{\varphi}^{\times}\right)^k\\
&= \exp\left(-\Delta t \boldsymbol{\varphi}^{\times}\right), && \text{(48)}\\
\mathbf{g}_{k-1}^2 &= \Delta t \mathbf{g} - \frac{1}{2}\Delta t^2 \boldsymbol{\varphi}^{\times}\mathbf{g} + \frac{1}{6}\Delta t^3\left(\boldsymbol{\varphi}^{\times}\right)^2\mathbf{g} - \frac{1}{24}\Delta t^4\left(\boldsymbol{\varphi}^{\times}\right)^3\mathbf{g} + \dots\\
&= \Delta t\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\left(-\Delta t \boldsymbol{\varphi}^{\times}\right)^k\right)\mathbf{g}\\
&= \Delta t \mathbf{J}_{\ell}\left(-\Delta t \boldsymbol{\varphi}\right)\mathbf{g}, && \text{(49)}\\
\mathbf{g}_{k-1}^3 &= -\frac{1}{2}\Delta t^2\mathbf{g} + \frac{1}{6}\Delta t^3\boldsymbol{\varphi}^{\times}\mathbf{g} - \frac{1}{24}\Delta t^4\left(\boldsymbol{\varphi}^{\times}\right)^2\mathbf{g} + \dots\\
&= -\frac{1}{2}\Delta t^2\left(\mathbf{1} - \frac{1}{3}\left(\Delta t \boldsymbol{\varphi}^{\times}\right) + \frac{1}{12}\left(\Delta t \boldsymbol{\varphi}^{\times}\right)^2 - \dots\right)\mathbf{g}\\
&= -\frac{1}{2}\Delta t^2\left(2\sum_{k=0}^{\infty}\frac{1}{(k+2)!}\left(-\Delta t \boldsymbol{\varphi}^{\times}\right)^k\right)\mathbf{g}\\
&= -\tfrac{1}{2}\Delta t^2\mathbf{N}\left(-\Delta t \boldsymbol{\varphi}\right)\mathbf{g}, && \text{(50)}
\end{aligned}
$$

where the expression for $\mathbf{N}$ is in [10, (9.249b)], Thus, (47) can be written as

$$
\mathbf{G}_{k-1} = \begin{bmatrix} \exp\left(-\Delta t \boldsymbol{\varphi}^{\times}\right) & \Delta t \mathbf{J}_{\ell}\left(-\Delta t \boldsymbol{\varphi}\right)\mathbf{g} & -\tfrac{1}{2}\Delta t^2 \mathbf{N}\left(-\Delta t \boldsymbol{\varphi}\right)\mathbf{g} \\ \mathbf{0} & 1 & -\Delta t \\ \mathbf{0} & 0 & 1 \end{bmatrix}. \tag{51}
$$

B Discrete-time Input Matrix

The input matrix $\mathbf{U}(\cdot)$ is time variant but kept constant over the interval $[t_{k-1}, t_k]$. Based on the assumption, the derivation of the discrete-time input matrix is analogous to the discrete-time gravity matrix,

$$\begin{aligned}\mathbf{U}_{k-1} &= \exp\left(\Delta t \mathbf{U}(t_{k-1})\right) && (52a)\\ &= \begin{bmatrix} \exp\left(\Delta t \boldsymbol{\omega}_{k-1}^{\times}\right) & \Delta t \mathbf{J}_\ell\left(\Delta t \boldsymbol{\omega}_{k-1}\right)\mathbf{f}_{k-1} & \frac{1}{2}\Delta t^2 \mathbf{N}\left(\Delta t \boldsymbol{\omega}_{k-1}\right)\mathbf{f}_{k-1} \\ \mathbf{0} & 1 & \Delta t \\ \mathbf{0} & 0 & 1 \end{bmatrix}. && (52b)\end{aligned}$$

C Preintegrated Gravity Matrix

Preintegration value $\mathbf{G}_{ij}$ from (24) is given by

$$\begin{aligned}\mathbf{G}_{ij} &= \prod_{k=i}^{j-1} \mathbf{G}_k && (53a)\\ &= \prod_{k=i}^{j-1} \begin{bmatrix} \mathrm{Exp}\left(-\Delta t_{kk+1}\boldsymbol{\varphi}\right) & \Delta t_{kk+1}\mathbf{J}_\ell\left(-\Delta t_{kk+1}\boldsymbol{\varphi}\right)\mathbf{g} & -\frac{1}{2}\Delta t_{kk+1}^2 \mathbf{N}\left(-\Delta t_{kk+1}\boldsymbol{\varphi}\right)\mathbf{g} \\ \mathbf{0} & 1 & -\Delta t_{kk+1} \\ \mathbf{0} & 0 & 1 \end{bmatrix}, && (53b)\end{aligned}$$

where $\Delta t_{kk+1} = t_{k+1} - t_k$. Note that the gravity matrix is composed on the left, i.e. for $i = 1$ and $j = 3$, the gravity matrix is

$$\begin{aligned}\mathbf{G}_{13} &= \mathbf{G}_2 \circ \mathbf{G}_1 && (54a)\\ &= \begin{bmatrix} \mathrm{Exp}\left(-\Delta t_{23}\boldsymbol{\varphi}\right) & \Delta t_{23}\mathbf{J}_\ell\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\mathbf{g} & -\frac{1}{2}\Delta t_{23}^2 \mathbf{N}\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\mathbf{g} \\ \mathbf{0} & 1 & -\Delta t_{23} \\ \mathbf{0} & 0 & 1 \end{bmatrix} \\ &\quad \begin{bmatrix} \mathrm{Exp}\left(-\Delta t_{12}\boldsymbol{\varphi}\right) & \Delta t_{12}\mathbf{J}_\ell\left(-\Delta t_{12}\boldsymbol{\varphi}\right)\mathbf{g} & -\frac{1}{2}\Delta t_{12}^2 \mathbf{N}\left(-\Delta t_{12}\boldsymbol{\varphi}\right)\mathbf{g} \\ \mathbf{0} & 1 & -\Delta t_{12} \\ \mathbf{0} & 0 & 1 \end{bmatrix} && (54b)\\ &= \begin{bmatrix} \mathbf{G}_{13}^1 & \mathbf{g}_{13}^2 & \mathbf{g}_{13}^3 \\ \mathbf{0} & 1 & -\Delta t_1 - \Delta t_2 \\ \mathbf{0} & 0 & 1 \end{bmatrix}, && (54c)\end{aligned}$$

where

$$\begin{aligned}\mathbf{G}_{13}^1 &= \mathrm{Exp}\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\mathrm{Exp}\left(-\Delta t_{12}\boldsymbol{\varphi}\right), && (55)\\ \mathbf{g}_{13}^2 &= \mathrm{Exp}\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\Delta t_{12}\mathbf{J}_\ell\left(-\Delta t_{12}\boldsymbol{\varphi}\right)\mathbf{g} + \Delta t_{23}\mathbf{J}_\ell\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\mathbf{g}, && (56)\\ \mathbf{g}_{13}^3 &= -\mathrm{Exp}\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\tfrac{1}{2}\Delta t_1^2 \mathbf{N}\left(-\Delta t_{12}\boldsymbol{\varphi}\right)\mathbf{g} - \Delta t_{12}\Delta t_{23}\mathbf{J}_\ell\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\mathbf{g} - \tfrac{1}{2}\Delta t_{23}^2 \mathbf{N}\left(-\Delta t_{23}\boldsymbol{\varphi}\right)\mathbf{g}. && (57)\end{aligned}$$

Note that if $\boldsymbol{\varphi} = \mathbf{0}$, then $\mathbf{G}_{13}^1 = \mathbf{1}$, $\mathbf{g}_{13}^2 = \left(\Delta t_{12} + \Delta t_{23}\right)\mathbf{g}$, and $\mathbf{g}_{13}^3 = -\frac{1}{2}\left(\Delta t_{12} + \Delta t_{23}\right)^2 \mathbf{g}$, then

$$\mathbf{G}_{13} = \begin{bmatrix} \mathbf{1} & \Delta t_{13}\mathbf{g} & -\frac{1}{2}\Delta t_{13}^2 \mathbf{g} \\ \mathbf{0} & 1 & -\Delta t_{13} \\ \mathbf{0} & 0 & 1 \end{bmatrix}. \tag{58}$$

D Adjoint Matrix of a Time Machine Matrix

The adjoint matrix of the time machine matrix $\mathbf{D}$ in (21) is defined as

$$\mathbf{ad}\left(\mathbf{D}\right)\boldsymbol{\xi} \triangleq \left(\mathbf{D}\boldsymbol{\xi}^\wedge - \boldsymbol{\xi}^\wedge \mathbf{D}\right)^\vee, \tag{59}$$

where $\boldsymbol{\xi}^\wedge \in \mathfrak{se}_2(3)$. This is not equal to the adjoint matrix of a Lie group since the time machine matrix and $\boldsymbol{\xi}^\wedge$ are not in the same Lie algebra. However, the adjoint matrix of the time machine matrix is useful in deriving the Jacobian matrices of functions acting on the time machine matrix. Also, note that this definition is only valid when the interacting element is $\boldsymbol{\xi}^\wedge \in \mathfrak{se}_2(3)$. The right-hand side of (59) evaluates to

$$\begin{aligned}
\left(\mathbf{D}\boldsymbol{\xi}^\wedge - \boldsymbol{\xi}^\wedge\mathbf{D}\right)^\vee &= \left(\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 0 & -1\\ \mathbf{0} & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_1^\times & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3\\ \mathbf{0} & 1 & 0\\ \mathbf{0} & 0 & 1\end{bmatrix} - \begin{bmatrix}\boldsymbol{\xi}_1^\times & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3\\ \mathbf{0} & 1 & 0\\ \mathbf{0} & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 0 & -1\\ \mathbf{0} & 0 & 0\end{bmatrix}\right)^\vee && \text{(60a)}\\
&= \left(\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 0 & -1\\ \mathbf{0} & 0 & 0\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{0} & -\boldsymbol{\xi}_2\\ \mathbf{0} & 0 & -1\\ \mathbf{0} & 0 & 0\end{bmatrix}\right)^\vee && \text{(60b)}\\
&= \begin{bmatrix}\mathbf{0} & \mathbf{0} & -\boldsymbol{\xi}_2\\ \mathbf{0} & 0 & 0\\ \mathbf{0} & 0 & 0\end{bmatrix}^\vee && \text{(60c)}\\
&= \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ -\boldsymbol{\xi}_2\end{bmatrix}. && \text{(60d)}
\end{aligned}$$

Thus, the left-hand side of (59) becomes

$$\mathbf{ad}\left(\mathbf{D}\right)\boldsymbol{\xi} = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{1} & \mathbf{0}\end{bmatrix}\boldsymbol{\xi} \tag{61}$$

to satisfy the definition of the adjoint.

The adjoint matrix of the discrete time-machine matrix is defined as

$$\mathbf{Ad}\left(\mathbf{D}_{k-1}\right)\boldsymbol{\xi} \triangleq \left(\mathbf{D}_{k-1}\boldsymbol{\xi}_k^\wedge\mathbf{D}_{k-1}^{-1}\right)^\vee. \tag{62}$$

Again, the definition of the adjoint matrix of the discrete time-machine matrix is not equal to the adjoint matrix of a Lie group since the discrete time-machine matrix $\mathbf{D}_k$ and $\exp(\boldsymbol{\xi}_k^\wedge) \in SE_2(3)$ are not in the same Lie group. The right-hand side of (62) evaluates to

$$\begin{aligned}
\left(\mathbf{D}_{k-1}\boldsymbol{\xi}_k^\wedge\mathbf{D}_{k-1}^{-1}\right)^\vee &= \left(\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1 & \Delta t\\ \mathbf{0} & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_1^\times & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3\\ \mathbf{0} & 1 & 0\\ \mathbf{0} & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1 & -\Delta t\\ \mathbf{0} & 0 & 1\end{bmatrix}\right)^\vee && \text{(63a)}\\
&= \left(\begin{bmatrix}\boldsymbol{\xi}_1^\times & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3\\ \mathbf{0} & 1 & \Delta t\\ \mathbf{0} & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1 & -\Delta t\\ \mathbf{0} & 0 & \end{bmatrix}\right)^\vee && \text{(63b)}\\
&= \left(\begin{bmatrix}\boldsymbol{\xi}_1^\times & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3 - \boldsymbol{\xi}_2\Delta t\\ \mathbf{0} & 1 & 0\\ \mathbf{0} & 0 & 1\end{bmatrix}\right)^\vee && \text{(63c)}\\
&= \begin{bmatrix}\boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\\ \boldsymbol{\xi}_3 - \boldsymbol{\xi}_2\Delta t\end{bmatrix}. && \text{(63d)}
\end{aligned}$$

Thus, to satisfy the definition of the adjoint, the left-hand side of (62) becomes

$$\mathbf{Ad}\left(\mathbf{D}_{k-1}\right)\boldsymbol{\xi} = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & -\Delta t\mathbf{1} & \mathbf{1}\end{bmatrix}\boldsymbol{\xi}. \tag{64}$$

This also satisfies the property of the adjoint matrix, i.e.

$$\mathbf{Ad}\left(\mathbf{D}_{k-1}\right) = \exp\left(\mathbf{ad}\left(\Delta t\mathbf{D}\right)\right). \tag{65}$$

E IMU Factor

Two versions of the IMU factor are derived in this section: the preintegrated IMU factor and the continuous-time IMU factor.

E.1 Preintegrated IMU Factor

Recall the preintegrated IMU factor from (39a),

$$\mathbf{e}^I_{ij}(\mathcal{X}) = \begin{bmatrix} {}^{(\cdot)}\left(\tilde{\mathbf{U}}^+_{ij} \ominus \check{\mathbf{U}}_{ij}\left(\mathbf{X}_i, \mathbf{X}_j\right)\right) \\ \boldsymbol{\beta}_j - \boldsymbol{\beta}_i \end{bmatrix}, \tag{66a}$$

$$\tilde{\mathbf{U}}^+_{ij} = \tilde{\mathbf{U}}_{ij} \oplus^{(\cdot)} \frac{D\mathbf{U}_{ij}}{D\boldsymbol{\beta}_i}\delta\boldsymbol{\beta}_i, \tag{66b}$$

where $(\cdot)$ is either R or L that denotes the right- or left-operators, respectively. The Jacobian of the preintegrated IMU factor w.r.t. the states at time t_i and t_j are computed using a chain rule,

$$\frac{{}^{(\cdot)}D\mathbf{e}^I_{ij}}{D\mathbf{X}_i} = \begin{bmatrix} \frac{{}^{(\cdot)}D\mathbf{e}^I_{ij}}{D\check{\mathbf{U}}_{ij}} \frac{{}^{(\cdot)}D\check{\mathbf{U}}_{ij}}{D\mathbf{X}_i} \\ \mathbf{0} \end{bmatrix}, \qquad \frac{{}^{(\cdot)}D\mathbf{e}^I_{ij}}{D\boldsymbol{\beta}_i} = \begin{bmatrix} \frac{{}^{(\cdot)}D\mathbf{e}^I_{ij}}{D\tilde{\mathbf{U}}^+_{ij}} \frac{{}^{(\cdot)}D\tilde{\mathbf{U}}^+_{ij}}{D\boldsymbol{\beta}_i} \\ -\mathbf{1} \end{bmatrix}, \tag{67a}$$

$$\frac{{}^{(\cdot)}D\mathbf{e}^I_{ij}}{D\mathbf{X}_j} = \begin{bmatrix} \frac{{}^{(\cdot)}D\mathbf{e}^I_{ij}}{D\check{\mathbf{U}}_{ij}} \frac{{}^{(\cdot)}D\check{\mathbf{U}}_{ij}}{D\mathbf{X}_j} \\ \mathbf{0} \end{bmatrix}, \qquad \frac{D\mathbf{e}^I_{ij}}{D\boldsymbol{\beta}_j} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}. \tag{67b}$$

Note that they both share the Jacobian of the residual w.r.t. the predicted RMI, $\check{\mathbf{U}}_{ij}$,

$$\frac{{}^{R}D\mathbf{e}^I_{ij}}{D\check{\mathbf{U}}_{ij}} = -\mathbf{J}_r^{-1}\left(\bar{\mathbf{e}}\right)\mathbf{Ad}\left(\operatorname{Exp}\left(-\bar{\mathbf{e}}\right)\right), \qquad \frac{{}^{L}D\mathbf{e}^I_{ij}}{D\check{\mathbf{U}}_{ij}} = -\mathbf{J}_\ell^{-1}\left(\bar{\mathbf{e}}\right)\mathbf{Ad}\left(\operatorname{Exp}\left(\bar{\mathbf{e}}\right)\right), \tag{68}$$

where $\bar{\mathbf{e}} = \mathbf{e}^I_{ij}(\bar{\mathcal{X}})$. Jacobian of the predicted RMI w.r.t. the extended pose at t_i is given as

$$\frac{{}^{R}D\check{\mathbf{U}}_{ij}}{D\mathbf{X}_i} = -\mathbf{Ad}\left(\check{\mathbf{U}}^{-1}_{ij}\right) = \mathbf{Ad}\left(\check{\mathbf{U}'}^{-1}_{ij}\right)\mathbf{Ad}\left(\mathbf{D}^{-1}_{ij}\right), \qquad \frac{{}^{L}D\check{\mathbf{U}}_{ij}}{D\mathbf{X}_i} = -\mathbf{Ad}\left(\mathbf{X}^{-1}_i\right). \tag{69}$$

Jacobian of the residual w.r.t. the extended pose at t_j is given as

$$\frac{{}^{R}D\check{\mathbf{U}}_{ij}}{D\mathbf{X}_j} = \mathbf{1}, \qquad \frac{{}^{L}D\check{\mathbf{U}}_{ij}}{D\mathbf{X}_j} = \mathbf{Ad}\left(\mathbf{X}^{-1}_i{\mathbf{G}'}^{-1}_{ij}\right)\mathbf{Ad}\left(\mathbf{D}_{ij}\right). \tag{70}$$

Moreover, Jacobian of the residual w.r.t. the bias corrected RMI, $\tilde{\mathbf{U}}^+_{ij}$, is given as

$$\frac{{}^{R}D\mathbf{e}^I_{ij}}{D\tilde{\mathbf{U}}^+_{ij}} = \mathbf{J}_r^{-1}\left(\bar{\mathbf{e}}\right), \qquad \frac{{}^{L}D\mathbf{e}^I_{ij}}{D\tilde{\mathbf{U}}^+_{ij}} = \mathbf{J}_\ell^{-1}\left(\bar{\mathbf{e}}\right). \tag{71}$$

And, Jacobian of the bias corrected RMI w.r.t. the IMU bias at t_i is given as

$$\frac{{}^{R}D\tilde{\mathbf{U}}^+_{ij}}{D\boldsymbol{\beta}_i} = \mathbf{J}_r\left(\frac{D\tilde{\mathbf{U}}_{ij}}{D\boldsymbol{\beta}_i}(\boldsymbol{\beta}_i - \bar{\boldsymbol{\beta}}_i)\right)\frac{D\tilde{\mathbf{U}}_{ij}}{D\boldsymbol{\beta}_i}, \qquad \frac{{}^{L}D\tilde{\mathbf{U}}^+_{ij}}{D\boldsymbol{\beta}_i} = \mathbf{J}_\ell\left(\frac{D\tilde{\mathbf{U}}_{ij}}{D\boldsymbol{\beta}_i}(\boldsymbol{\beta}_i - \bar{\boldsymbol{\beta}}_i)\right)\frac{D\tilde{\mathbf{U}}_{ij}}{D\boldsymbol{\beta}_i}. \tag{72}$$

E.2 Gyro as Measurement Factor

Before deriving the continuous-time IMU factor, recall the bias interpolation function from (41),

$$\boldsymbol{\beta}(\tau) = (1-\alpha)\boldsymbol{\beta}_i + \alpha\boldsymbol{\beta}_j, \qquad \alpha = \frac{\tau - t_i}{t_j - t_i},\ t_i \le \tau \le t_j.$$

The Jacobian of the linear interpolation w.r.t. the two knot states is given as

$$\frac{D\boldsymbol{\beta}(\tau)}{D\boldsymbol{\beta}_i} = \frac{D\boldsymbol{\beta}_\tau}{D\boldsymbol{\beta}_i} = 1 - \alpha, \tag{73a}$$

$$\frac{D\boldsymbol{\beta}(\tau)}{D\boldsymbol{\beta}_j} = \frac{D\boldsymbol{\beta}_\tau}{D\boldsymbol{\beta}_j} = \alpha. \tag{73b}$$

Now, recall the gyroscope measurement factor from (40),

$$\mathbf{e}_\tau^{Ig}(\mathcal{X}) = \tilde{\boldsymbol{\omega}}_{b_\tau}^{b_\tau a} - \boldsymbol{\omega}_b^{ba}(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau) - \boldsymbol{\beta}_b^g(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau), \qquad t_i < \tau < t_j.$$

Given $\boldsymbol{\omega}_\tau = \boldsymbol{\omega}_b^{ba}(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau)$ and $\boldsymbol{\beta}_\tau^g = \boldsymbol{\beta}_b^g(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j, \tau)$, the Jacobian of the interpolated angular velocity and bias w.r.t. the $\mathbf{T}_\tau$, $\boldsymbol{\varpi}_\tau$, and $\boldsymbol{\beta}_\tau$ are given as

$$\frac{D\boldsymbol{\omega}_\tau}{D\mathbf{T}_\tau} = \mathbf{0}, \qquad \frac{D\boldsymbol{\omega}_\tau}{D\boldsymbol{\varpi}_\tau} = \begin{bmatrix}\mathbf{1} & \mathbf{0}\end{bmatrix}, \qquad \frac{D\boldsymbol{\omega}_\tau}{D\boldsymbol{\beta}_\tau} = \mathbf{0}, \tag{74a}$$

$$\frac{D\boldsymbol{\beta}_\tau^g}{D\mathbf{T}_\tau} = \mathbf{0}, \qquad \frac{D\boldsymbol{\beta}_\tau^g}{D\boldsymbol{\varpi}_\tau} = \mathbf{0}, \qquad \frac{D\boldsymbol{\beta}_\tau^g}{D\boldsymbol{\beta}_\tau} = \begin{bmatrix}\mathbf{1} & \mathbf{0}\end{bmatrix}. \tag{74b}$$

Then, the Jacobian of the residual w.r.t. the interpolated angular velocity and bias are given as

$$\frac{{}^{(\cdot)}D\mathbf{e}_\tau^{Ig}}{D\mathbf{X}_k} = -\frac{D\boldsymbol{\omega}_\tau}{D\boldsymbol{\varpi}_\tau}\frac{{}^{(\cdot)}D\boldsymbol{\varpi}_\tau}{D\mathbf{X}_k}, \qquad \frac{D\mathbf{e}_\tau^{Ig}}{D\boldsymbol{\omega}_k} = -\frac{D\boldsymbol{\omega}_\tau}{D\boldsymbol{\varpi}_\tau}\frac{D\boldsymbol{\varpi}_\tau}{D\boldsymbol{\omega}_k}, \qquad \frac{D\mathbf{e}_\tau^{Ig}}{D\boldsymbol{\beta}_k} = -\frac{D\boldsymbol{\beta}_\tau^g}{D\boldsymbol{\beta}_\tau}\frac{D\boldsymbol{\beta}_\tau}{D\boldsymbol{\beta}_k}, \tag{75}$$

where $k \in \{i, j\}$, and ${}^{(\cdot)}D\boldsymbol{\varpi}_\tau / D\mathbf{X}_k$ and $D\boldsymbol{\varpi}_\tau / D\boldsymbol{\omega}_k$ are the interpolation Jacobians.

E.3 Preintegrated Velocity Factor

Recall the accelerometer measurement factor from (44),

$$\mathbf{e}_{ij}^{Ia}(\mathcal{X}) = \begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix}{}^{(\cdot)}\left(\mathbf{U}'_{ij} \ominus \check{\mathbf{U}}'_{ij}\right),$$
$$\check{\mathbf{U}}'_{ij} = \left(\mathbf{G}'_{ij}\mathbf{T}'_i\right)^{-1}\mathbf{T}'_j.$$

The Jacobians of the accelerometer measurement residual w.r.t. the state are

$$\frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\mathbf{X}_k} = \begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix}\left(\frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\mathbf{U}'_{ij}}\frac{D\mathbf{U}'_{ij}}{D\mathbf{X}_k} + \frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\check{\mathbf{U}}'_{ij}}\frac{D\check{\mathbf{U}}'_{ij}}{D\mathbf{X}_k}\right), \tag{77a}$$

$$\frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\boldsymbol{\omega}_k} = \begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix}\frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\mathbf{U}'_{ij}}\frac{D\mathbf{U}'_{ij}}{D\boldsymbol{\omega}_k}, \tag{77b}$$

$$\frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\boldsymbol{\beta}_k} = \begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix}\frac{{}^{(\cdot)}D\mathbf{e}_{ij}^{Ia}}{D\mathbf{U}'_{ij}}\frac{D\mathbf{U}'_{ij}}{D\boldsymbol{\beta}_k}, \tag{77c}$$

where $k \in \{i, j\}$. Then, the Jacobian of the residual w.r.t. the preintegrated and predicted RMI, $\check{\mathbf{U}}'_{ij}$ and $\mathbf{U}'_{ij}$, is respectively given as

$$\frac{{}^{R}D\mathbf{e}_{ij}^{Ia}}{D\check{\mathbf{U}}'_{ij}} = -\mathbf{J}_r^{-1}(\bar{\mathbf{e}})\,\mathbf{Ad}\left(\operatorname{Exp}(-\bar{\mathbf{e}})\right), \qquad \frac{{}^{L}D\mathbf{e}_{ij}^{Ia}}{D\check{\mathbf{U}}'_{ij}} = -\mathbf{J}_\ell^{-1}(\bar{\mathbf{e}})\,\mathbf{Ad}\left(\operatorname{Exp}(\bar{\mathbf{e}})\right), \tag{78a}$$

$$\frac{{}^{R}D\mathbf{e}_{ij}^{Ia}}{D\mathbf{U}'_{ij}} = \mathbf{J}_r^{-1}(\bar{\mathbf{e}}), \qquad \frac{{}^{L}D\mathbf{e}_{ij}^{Ia}}{D\mathbf{U}'_{ij}} = \mathbf{J}_\ell^{-1}(\bar{\mathbf{e}}), \tag{78b}$$

where $\bar{\mathbf{e}} = \mathbf{e}_{ij}^{Ia}(\bar{\mathcal{X}})$. The Jacobians of the predicted RMI w.r.t. the state at time t_i and t_j are

$$\frac{{}^{(\cdot)}D\check{\mathbf{U}}'_{ij}}{D\mathbf{X}_i} = \begin{bmatrix}\frac{{}^{(\cdot)}D\check{\mathbf{U}}'_{ij}}{D\mathbf{T}'_i} & \mathbf{0}\end{bmatrix}, \qquad \frac{{}^{R}D\check{\mathbf{U}}'_{ij}}{D\mathbf{T}'_i} = -\mathbf{Ad}\left(\bar{\check{\mathbf{U}}}'^{-1}_{ij}\right), \qquad \frac{{}^{L}D\check{\mathbf{U}}'_{ij}}{D\mathbf{T}'_i} = -\mathbf{Ad}\left(\bar{\mathbf{T}}'^{-1}_i\right), \tag{79a}$$

$$\frac{{}^{(\cdot)}D\check{\mathbf{U}}'_{ij}}{D\mathbf{X}_j} = \begin{bmatrix}\frac{{}^{(\cdot)}D\check{\mathbf{U}}'_{ij}}{D\mathbf{T}'_j} & \mathbf{0}\end{bmatrix}, \qquad \frac{{}^{R}D\check{\mathbf{U}}'_{ij}}{D\mathbf{T}'_j} = \mathbf{1}, \qquad \frac{{}^{L}D\check{\mathbf{U}}'_{ij}}{D\mathbf{T}'_j} = \mathbf{Ad}\left(\left(\mathbf{G}'_{ij}\mathbf{T}'_i\right)^{-1}\right). \tag{79b}$$

Now, to derive the Jacobian of the preintegration w.r.t. the state, $D\mathbf{U}'_{ij}/D\boldsymbol{\mathcal{X}}_i$, the Jacobian w.r.t. the state at each time step is first derived. Consider the discrete-time input composition

$$\mathbf{U}'_{i\tau+1} = \mathbf{U}'_{i\tau}\mathrm{Exp}\left(\mathbf{u}'_{\tau}\delta t\right), \qquad t_i \leq \tau < t_j. \tag{80}$$

The discrete-time input composition is linearized with right-perturbation as

$$\begin{aligned}
\mathbf{U}'_{i\tau+1} &= \mathbf{U}'_{i\tau}\mathrm{Exp}\left(\mathbf{u}'_{\tau}\delta t\right), && \text{(81a)}\\
\bar{\mathbf{U}}'_{i\tau+1}\delta\mathbf{U}'_{i\tau+1} &= \bar{\mathbf{U}}'_{i\tau}\delta\mathbf{U}'_{i\tau}\mathrm{Exp}\left(\left(\bar{\mathbf{u}}'_{\tau}+\delta\mathbf{u}'_{\tau}\right)\delta t\right) && \text{(81b)}\\
&\approx \bar{\mathbf{U}}'_{i\tau}\delta\mathbf{U}'_{i\tau}\mathrm{Exp}\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\mathrm{Exp}\left(\mathbf{J}_r\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}\right) && \text{(81c)}\\
&= \bar{\mathbf{U}}'_{i\tau}\mathrm{Exp}\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\mathrm{Exp}\left(\mathbf{Ad}\left(\bar{\mathbf{U}}'^{-1}_{\tau}\right)\delta\mathbf{u}'_{i\tau}\right)\mathrm{Exp}\left(\mathbf{J}_r\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}\right), && \text{(81d)}\\
\delta\mathbf{U}'_{i\tau+1} &= \mathrm{Exp}\left(\mathbf{Ad}\left(\bar{\mathbf{U}}'^{-1}_{\tau}\right)\delta\mathbf{u}'_{i\tau}\right)\mathrm{Exp}\left(\mathbf{J}_r\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}\right), && \text{(81e)}\\
\delta\mathbf{u}'_{i\tau+1} &= \mathbf{Ad}\left(\bar{\mathbf{U}}'^{-1}_{\tau}\right)\delta\mathbf{u}'_{i\tau} + \mathbf{J}_r\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}. && \text{(81f)}
\end{aligned}$$

Similarly, with left-perturbation, the discrete-time input composition is linearized as

$$\begin{aligned}
\mathbf{U}'_{i\tau+1} &= \mathbf{U}'_{i\tau}\mathrm{Exp}\left(\mathbf{u}'_{\tau}\delta t\right), && \text{(82a)}\\
\delta\mathbf{U}'_{i\tau+1}\bar{\mathbf{U}}'_{i\tau+1} &= \delta\mathbf{U}'_{i\tau}\bar{\mathbf{U}}'_{i\tau}\mathrm{Exp}\left(\left(\bar{\mathbf{u}}'_{\tau}+\delta\mathbf{u}'_{\tau}\right)\delta t\right) && \text{(82b)}\\
&\approx \delta\mathbf{U}'_{i\tau}\bar{\mathbf{U}}'_{i\tau}\mathrm{Exp}\left(\mathbf{J}_\ell\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}\right)\mathrm{Exp}\left(\bar{\mathbf{u}}'_{\tau}\delta t\right) && \text{(82c)}\\
&= \delta\mathbf{U}'_{i\tau}\mathrm{Exp}\left(\mathbf{Ad}\left(\bar{\mathbf{U}}'_{i\tau}\right)\mathbf{J}_\ell\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}\right)\bar{\mathbf{U}}'_{i\tau}\mathrm{Exp}\left(\bar{\mathbf{u}}'_{\tau}\delta t\right), && \text{(82d)}\\
\delta\mathbf{U}'_{i\tau+1} &= \delta\mathbf{U}'_{i\tau}\mathrm{Exp}\left(\mathbf{Ad}\left(\bar{\mathbf{U}}'_{i\tau}\right)\mathbf{J}_\ell\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}\right), && \text{(82e)}\\
\delta\mathbf{u}'_{i\tau+1} &= \delta\mathbf{u}'_{i\tau} + \mathbf{Ad}\left(\bar{\mathbf{U}}'_{i\tau}\right)\mathbf{J}_\ell\left(\bar{\mathbf{u}}'_{\tau}\delta t\right)\delta t\delta\mathbf{u}'_{\tau}. && \text{(82f)}
\end{aligned}$$

Thus, the Jacobian of the preintegration w.r.t. the state at each time step can be written as

$$\frac{{}^{R}D\mathbf{U}'_{ij}}{D\mathbf{u}'_{\tau}} = \mathbf{Ad}\left(\bar{\mathbf{U}}'_{j-1\tau}\right)\mathbf{J}_r\left(\mathbf{u}_{\tau}\delta t\right)\delta t, \qquad \frac{{}^{L}D\mathbf{U}'_{ij}}{D\mathbf{u}'_{\tau}} = \mathbf{Ad}\left(\bar{\mathbf{U}}'_{i\tau}\right)\mathbf{J}_\ell\left(\mathbf{u}_{\tau}\delta t\right)\delta t. \tag{83}$$

The relation between the perturbed input, $\delta\mathbf{u}'_{\tau}$, and the states are

$$\delta\mathbf{u}'_{\tau} = \begin{bmatrix}\delta\boldsymbol{\omega}^{b_\tau a}_{b_\tau}\\ -\delta\boldsymbol{\beta}^{a}_{b_\tau}\end{bmatrix}. \tag{84}$$

To relate the Jacobian at each time step to the input states, $(\boldsymbol{\mathcal{X}}_i, \boldsymbol{\mathcal{X}}_j)$, the interpolation Jacobians can be chained to the above Jacobian,

$$\begin{aligned}
\delta\boldsymbol{\varpi}_{\tau} = \begin{bmatrix}\delta\boldsymbol{\omega}_{\tau}\\ \delta\mathbf{v}_{b_\tau}\end{bmatrix} &= \frac{{}^{(\cdot)}D\boldsymbol{\varpi}_{\tau}}{D\mathbf{X}_i}\delta\boldsymbol{\xi} + \frac{{}^{(\cdot)}D\boldsymbol{\varpi}_{\tau}}{D\boldsymbol{\omega}_i}\delta\boldsymbol{\omega}_i + \frac{{}^{(\cdot)}D\boldsymbol{\varpi}_{\tau}}{D\mathbf{X}_j}\delta\boldsymbol{\xi}_j + \frac{{}^{(\cdot)}D\boldsymbol{\varpi}_{\tau}}{D\boldsymbol{\omega}_j}\delta\boldsymbol{\omega}_j && \text{(85a)}\\
&= \begin{bmatrix}\frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\mathbf{X}_i}\\ \frac{{}^{(\cdot)}D\mathbf{v}_{b_\tau}}{D\mathbf{X}_i}\end{bmatrix}\delta\boldsymbol{\xi}_i + \begin{bmatrix}\frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\boldsymbol{\omega}_i}\\ \frac{{}^{(\cdot)}D\mathbf{v}_{b_\tau}}{D\boldsymbol{\omega}_i}\end{bmatrix}\delta\boldsymbol{\omega}_i + \begin{bmatrix}\frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\mathbf{X}_j}\\ \frac{{}^{(\cdot)}D\mathbf{v}_{b_\tau}}{D\mathbf{X}_j}\end{bmatrix}\delta\boldsymbol{\xi}_j + \begin{bmatrix}\frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\boldsymbol{\omega}_j}\\ \frac{{}^{(\cdot)}D\mathbf{v}_{b_\tau}}{D\boldsymbol{\omega}_j}\end{bmatrix}\delta\boldsymbol{\omega}_j, && \text{(85b)}\\
\delta\boldsymbol{\omega}_{\tau} &= \frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\mathbf{X}_i}\delta\boldsymbol{\xi}_i + \frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\boldsymbol{\omega}_i}\delta\boldsymbol{\omega}_i + \frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\mathbf{X}_j}\delta\boldsymbol{\xi}_j + \frac{{}^{(\cdot)}D\boldsymbol{\omega}_{\tau}}{D\boldsymbol{\omega}_j}\delta\boldsymbol{\omega}_j, && \text{(85c)}
\end{aligned}$$

where $D\boldsymbol{\varpi}_{\tau}/D\mathbf{X}_k$ and $D\boldsymbol{\varpi}_{\tau}/D\boldsymbol{\omega}_k$ are the interpolation Jacobians for $k \in \{i, j\}$. Thus, the perturbed input can be rewritten as

$$\frac{D\mathbf{u}'_{\tau}}{D\mathbf{X}_k} = \begin{bmatrix}\frac{D\boldsymbol{\omega}_{\tau}}{D\mathbf{X}_k}\\ \mathbf{0}\end{bmatrix}, \qquad \frac{D\mathbf{u}'_{\tau}}{D\boldsymbol{\omega}_k} = \begin{bmatrix}\frac{D\boldsymbol{\omega}_{\tau}}{D\boldsymbol{\omega}_k}\\ \mathbf{0}\end{bmatrix}, \qquad \frac{D\mathbf{u}'_{\tau}}{D\boldsymbol{\beta}_k} = \begin{bmatrix}\mathbf{0}\\ -\frac{D\boldsymbol{\beta}^{a}_{b_\tau}}{D\boldsymbol{\beta}_k}\end{bmatrix}, \tag{86}$$

where $k \in \{i, j\}$. Finally, the Jacobian of the preintegration w.r.t. the state can be written as the sum of the Jacobian and chained to the Jacobian of the preintegration w.r.t. the state at time at each time step

$$\frac{{}^{(\cdot)}D\mathbf{U}'_{ij}}{D\mathbf{X}_k} = \sum_{\tau=i}^{j-1} \frac{{}^{(\cdot)}D\mathbf{U}'_{ij}}{D\mathbf{u}'_\tau} \frac{D\mathbf{u}'_\tau}{D\mathbf{X}_k}, \tag{87a}$$

$$\frac{{}^{(\cdot)}D\mathbf{U}'_{ij}}{D\boldsymbol{\omega}_k} = \sum_{\tau=i}^{j-1} \frac{{}^{(\cdot)}D\mathbf{U}'_{ij}}{D\mathbf{u}'_\tau} \frac{D\mathbf{u}'_\tau}{D\boldsymbol{\omega}_k}, \tag{87b}$$

$$\frac{{}^{(\cdot)}D\mathbf{U}'_{ij}}{D\boldsymbol{\beta}_k} = \sum_{\tau=i}^{j-1} \frac{{}^{(\cdot)}D\mathbf{U}'_{ij}}{D\mathbf{u}'_\tau} \frac{D\mathbf{u}'_\tau}{D\boldsymbol{\omega}_k}. \tag{87c}$$

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